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A Critical Points Theorem and Nonlinear Differential Problems

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Abstract. The existence of two intervals of positive real parameters λ for which the functional $\Phi + \lambda\Psi$ has three critical points, whose norms are uniformly bounded in respect to λ belonging to one of the two intervals, is established. As an example of an application to nonlinear differential problems, a two point boundary value problem is considered and multiplicity results are obtained.

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1. Introduction

Very recently, in [3] and [10] two theorems on the existence of three critical points for functionals of the type

$$\Phi + \lambda\Psi$$

were obtained.

The first of the cited results (Theorem 1.1 below), under suitable assumptions, ensured the existence of an interval Λ_2 such that the previous functional admits at least three solutions whose norms are uniformly bounded in respect to λ; moreover, an upper bound for Λ_2 was established. This result was obtained as a consequence of the three critical points theorem of B. Ricceri in [20] by using some results on a suitable minimax inequality established in [10] (see also [6]). Applications to nonlinear differential problems of the three critical points theorem of B. Ricceri and its consequences were given in [5–11, 16, 20, 21] (see also [17] for the non-smooth case).

The second result (Theorem 1.2 below), under a different set of assumptions in respect to the previous one, established the existence of a precise open interval Λ_1 for which the previous functional has three critical points. Its proof, obtained with a completely different technique from the previous one, was based on the variational principle of Ricceri [19] and on the classical Mountain Pass Theorem by Pucci and

Serrin [18]. Applications of this result to two point boundary value problems were given in [3] and [4].

For the reader's convenience we recall here these three critical points theorems.

THEOREM 1.1 ([10, Theorem 2.1]). *Let X be a separable and reflexive real Banach space, and let* $\Phi, J : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals. Assume that there exists* $x_0 \in X$ *such that* $\Phi(x_0) = J(x_0) = 0$ *and* $\Phi(x) \geqslant 0$ *for every* $x \in X$ *and that there exist* $x_1 \in X$, $r > 0$ *such that*

(i) $r < \Phi(x_1)$

(ii) $\sup_{\Phi(x)<r} J(x) < r\dfrac{J(x_1)}{\Phi(x_1)}$.

Further, put

$$\overline{a} = \frac{hr}{r\frac{J(x_1)}{\Phi(x_1)} - \sup_{\Phi(x)<r} J(x)},$$

with $h > 1$, *assume that the functional* $\Phi - \lambda J$ *is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and*

(iii) $\lim_{\|x\|\to+\infty} \Big(\Phi(x) - \lambda J(x)\Big) = +\infty$, *for every* $\lambda \in [0, \overline{a}]$.

Then, there exists an open interval $\Lambda_2 \subseteq [0, \overline{a}]$ *and a positive real number* σ *such that, for each* $\lambda \in \Lambda_2$, *the equation*

$$\Phi'(x) - \lambda J'(x) = 0$$

admits at least three solutions in X whose norms are less than σ.

THEOREM 1.2 *([3, Theorem B]). Let X be a reflexive real Banach space*; $\Phi : X \longrightarrow \mathbb{R}$ *a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^*; $\Psi : X \longrightarrow \mathbb{R}$ *a continuosly Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*:

(i) $\lim_{\|x\|\to+\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty$ *for all* $\lambda \in [0, +\infty[$,

(ii) there is $r \in \mathbb{R}$ *such that*:

$$\inf_X \Phi < r,$$

and

$$\varphi_1(r) < \varphi_2(r),$$

where

$$\varphi_1(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w}\Psi}{r - \Phi(x)}, \tag{1.1}$$

$$\varphi_2(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \sup_{y\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)}, \tag{1.2}$$

and $\overline{\Phi^{-1}(] - \infty, r[)}^w$ *is the closure of* $\Phi^{-1}(] - \infty, r[)$ *in the weak topology.*
Then, for each $\lambda \in \Lambda_1 =]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$, *the equation*

$$\Phi'(x) + \lambda\Psi'(x) = 0$$

has at least three solutions in X.

The aim of this paper is to establish some conditions so that both previous theorems hold; thus, we have two intervals of parameters for which the functional $\Phi+\lambda\Psi$ has three critical points whose norms in respect to one of the two intervals satisfy a certain stability condition. The main result of the paper, Theorem 2.1, is presented in Section 2.

As an example of an application to nonlinear differential problems of Theorem 2.1, we consider the following two point boundary value problem

$$\begin{cases} -u'' = \lambda f(u) \\ u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

The main result in Section 3 is Theorem 3.1 that, under suitable assumptions, ensures the existence of two intervals Λ_1 and Λ_2 such that, for each $\lambda \in \Lambda_1 \cup \Lambda_2$, the problem (1.3) admits at least three classical solutions whose norms are uniformly bounded in respect to $\lambda \in \Lambda_2$.

Multiple solutions to problem (1.3) were obtained from several authors by using distinct techniques such as methods of lower and upper solutions or fixed points theorems (see, for instance, [1, 2, 12–15] and references therein). These results are mutually independent of ours; for example, in [12], the existence of $\lambda^* > 0$ such that for each positive $\lambda < \lambda^*$ the problem (1.3) admits two solutions is proved under a key assumption which is essentially opposite to one of our assumptions (see Remark 3.2 of [3]); and, moreover, we can apply our result even if the key assumption of Theorem 2 of [13] is not verified (see Remark 3.4 of [3]).

Finally, we present an example of application of Theorem 3.1 (Example 3.1) and one of its immediate consequences (Theorem 3.2).

2. The Main Result

In this Section we present the main result of the paper. Its proof is based on Theorem 1.1 and Theorem 1.2 in the Introduction.

THEOREM 3.1. *Let X be a separable and reflexive real Banach space*; $\Phi : X \longrightarrow \mathbb{R}$ *a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^*; $J : X \longrightarrow \mathbb{R}$ *a continuosly Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists* $x_0 \in X$ *such that* $\Phi(x_0) = J(x_0) = 0$ *and that*

$$\text{(i)} \quad \lim_{\|x\| \to +\infty} (\Phi(x) - \lambda J(x)) = +\infty \qquad \text{for all } \lambda \in [0, +\infty[.$$

Further, assume that there are $r > 0$, $x_1 \in X$ *such that*:

(ii) $r < \Phi(x_1)$

(iii) $\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x) < \frac{r}{r + \Phi(x_1)} J(x_1)$

Then, for each

$$\lambda \in \Lambda_1 = \left] \frac{\Phi(x_1)}{J(x_1) - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)}, \frac{r}{\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)} \right[,$$

the equation

$$\Phi'(x) - \lambda J'(x) = 0 \tag{2.1}$$

has at least three solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subseteq \left[0, \frac{hr}{r \frac{J(x_1)}{\Phi(x_1)} - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)} \right]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, the equation (2.1) *has at least three solutions in X whose norms are less than σ.*

Proof. From Theorem 1.1, taking into account that

$$\sup_{x \in \Phi^{-1}(]-\infty, r[)} J(x) \leqslant \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x) \quad \text{and} \quad \frac{r}{r + \Phi(x_1)} \leqslant \frac{r}{\Phi(x_1)},$$

we obtain that there exists an open interval

$$\Lambda_2 \subseteq \left[0, \frac{hr}{r \frac{J(x_1)}{\Phi(x_1)} - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)} \right]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, the equation (2.1) has at least three solutions in X whose norms are less than σ.

Moreover, putting $\Psi(x) = -J(x)$ for every $x \in X$, one has

$$\varphi_1(r) \leqslant \frac{\sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} J(x)}{r}$$

and

$$\varphi_2(r) \geqslant \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \Psi(x_1)}{\Phi(x_1) - \Phi(x)} \geqslant \frac{\inf_{x\in\Phi^{-1}(]-\infty,r[)} \Psi(x) - \Psi(x_1)}{\Phi(x_1)}$$

$$\geqslant \frac{J(x_1) - \sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} J(x)}{\Phi(x_1)},$$

where $\varphi_1(r)$ and $\varphi_2(r)$ are given by (1.1) and (1.2).
Therefore, from (iii) it follows that

$$\varphi_1(r) < \varphi_2(r).$$

Hence, from Theorem 1.2 one has that for each

$$\lambda \in \Lambda_1 = \left] \frac{\Phi(x_1)}{J(x_1) - \sup_{x\in\overline{\Phi^{-1}(]-\infty,r])}^{w}} J(x)}, \frac{r}{\sup_{x\in\overline{\Phi^{-1}(]-\infty,r])}^{w}} J(x)} \right[,$$

the equation (2.1) has at least three solutions in X and the proof is complete. □

REMARK 2.1. *Fixed* $h > 1$, *from*

$$0 < r < \Phi(x_1) \qquad \textit{and} \qquad \sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} J(x) < r\frac{J(x_1)}{\Phi(x_1)}$$

it follows that

$$\frac{hr}{r\frac{J(x_1)}{\Phi(x_1)} - \sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} J(x)} > \frac{\Phi(x_1)}{J(x_1) - \sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} J(x)}.$$

Therefore, the real intervals Λ_1 *and* Λ_2 *in Theorem 2.1 are such that either*

$$\Lambda_1 \cap \Lambda_2 = \emptyset$$

or

$$\Lambda_1 \cap \Lambda_2 \neq \emptyset.$$

In the first case, we actually obtain two distinct open intervals of positive real parameters for which Equation (2.1) *admits three solutions; in the second case, we obtain only one interval of positive real parameters, that is* $\Lambda_1 \cup \Lambda_2$, *for which the Equation* (2.1) *admits three solutions and, in addition, the subinterval* Λ_2 *for which the solutions are uniformly bounded.*

REMARK 2.2. *Theorem 1.1 (or, the more general Theorem 3 of [20]) and Theorem 1.2 were applied in several nonlinear differential problems to obtain multiple solutions (see [3–11, 16, 20, 21]). In a similar way, we can apply Theorem 2.1 to these nonlinear differential problems. In the next section, we will give an example of applications of Theorem 2.1 to a two point boundary value problem.*

3. Applications to Nonlinear Problems

Here, by way of example of applications to nonlinear differential problems of the three critical points theorem in Section 2, we consider the following two point boundary value problem

$$\begin{cases} -u'' = \lambda f(u) \\ u(0) = u(1) = 0. \end{cases} \tag{3.1}$$

We establish the following theorem.

THEOREM 3.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Put $F(t)=\int_0^t f(\xi)d\xi$ for every $t\in\mathbb{R}$ and assume that there exist four positive constants c, d, b, s, with $c<d$ and $s<2$, such that*

$$(i) \qquad F(t) < \tfrac{c^2}{2c^2+4d^2}\left(F(d) + \tfrac{1}{d}\int_0^d F(t)dt\right) \quad \textit{for every} \quad t\in[-c,c]$$

$$(ii) \qquad F(t) \leqslant b(1 + |t|^s) \quad \textit{for every} \quad t\in\mathbb{R}.$$

Then, for each

$$\lambda\in\Lambda_1 = \left]\frac{8d^2}{F(d) + \frac{1}{d}\int_0^d F(t)dt - 2\max\limits_{|t|\leqslant c}F(t)}, \frac{2c^2}{\max\limits_{|t|\leqslant c}F(t)}\right[,$$

the problem (3.1) *admits at least three solutions in* $C^2([0,1])$ *and, moreover, for each* $h>1$, *there exists an open interval*

$$\Lambda_2 \subseteq \left[0, \frac{8hd^2}{F(d) + \frac{1}{d}\int_0^d F(t)dt - \frac{4d^2}{c^2}\max\limits_{|t|\leqslant c}F(t)}\right]$$

such that, for each $\lambda\in\Lambda_2$, *the problem* (3.1) *admits at least three solutions in* $C^2([0,1])$ *whose norms in* $C^2([0,1])$ *are less than* σ.

Proof. Let X be the Sobolev space $W_0^{1,2}([0,1]$ endowed with the norm $\|x\| = \left(\int_0^1 |x'(t)|^2dt\right)^{\frac{1}{2}}$ and put $\Phi(x)=\frac{1}{2}\|x\|^2$, $J(x)=\int_0^1 F(x(t))dt$ for every $x\in X$.

It is well known that the critical points of the functional $\Phi - \lambda J$ in X are precisely the weak solutions of problem (3.1) and that the weak solutions, by using standard methods, belong to $C^2([0,1])$ and are classical solutions for problem (3.1). So, our end is to apply Theorem 2.1 to Φ and J.

Clearly, Φ is a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and J is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.

Furthermore, from (ii) we obtain

$$\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda J(x)) = +\infty$$

for all $\lambda \in [0, +\infty[$.

Now, put

$$x_1(t) := \begin{cases} 4dt & \text{if } t \in [0, \frac{1}{4}[\\ d & \text{if } t \in [\frac{1}{4}, \frac{3}{4}] \\ 4d(1 - t) & \text{if } t \in]\frac{3}{4}, 1] \end{cases}$$

and $r = 2c^2$. Clearly, one has $\Phi(x_1) = 4d^2$ and $J(x_1) = \frac{1}{2}F(d) + \frac{1}{2d}\int_0^d F(t)\mathrm{d}t$. So, since $c < d$, we obtain

$$r < \Phi(x_1).$$

Finally, taking into account that

$$\max_{t\in[0,1]}|x(t)| \leqslant \frac{1}{2}\|x\|$$

for all $x \in X$, one has

$$\sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(x) = \sup_{x\in\Phi^{-1}(]-\infty,r])} J(x) \leqslant \max_{|t|\leqslant c} F(t)$$

and moreover, one has

$$\frac{r}{r + \Phi(x_1)} J(x_1) = \frac{2c^2}{2c^2 + 4d^2}\left(\frac{1}{2}F(d) + \frac{1}{2d}\int_0^d F(t)\mathrm{d}t\right)$$

Therefore, from (i) one has

$$\sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(x) < \frac{r}{r + \Phi(x_1)} J(x_1).$$

Now, we can apply Theorem 2.1. Taking into account that

$$\frac{\Phi(x_1)}{J(x_1) - \sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(x)} \leqslant \frac{8d^2}{F(d) + \frac{1}{d}\int_0^d F(t)\mathrm{d}t - 2\max_{|t|\leqslant c} F(t)};$$

$$\frac{r}{\sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(x)} \geqslant \frac{2c^2}{\max_{|t|\leqslant c} F(t)};$$

$$\frac{hr}{r\frac{J(x_1)}{\Phi(x_1)} - \sup\limits_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} J(x)} \leqslant \frac{8hd^2}{F(d) + \frac{1}{d}\int_0^d F(t)\mathrm{d}t - \frac{4d^2}{c^2}\max\limits_{|t|\leqslant c} F(t)} = a;$$

from Theorem 2.1 it follows that, for each $\lambda\in\Lambda_1$, problem (3.1) admits at least three classical solutions and there exists an open interval $\Lambda_2\subseteq[0,a]$ and a real positive number $\overline{\sigma}$ such that, for each $\lambda\in\Lambda_2$, problem (3.1) admits at least three solutions in $C^2([0,1])$ whose norms in $W_0^{1,2}([0,1])$ are less than $\overline{\sigma}$. Hence, by choosing

$$\sigma > \max\left\{\frac{\overline{\sigma}}{2}, a \sup_{|t|\leqslant\frac{\overline{\sigma}}{2}} |f(t)|\right\},$$

we have the conclusion. □

REMARK 3.1. *Clearly, the conditions*

(k) $\int_0^d F(t)dt \geqslant 0$

(kk) $F(t) < \frac{1}{6}(\frac{c}{d})^2 F(d)$ *for every* $t\in[-c,c]$

imply (i) of Theorem 3.1.

REMARK 3.2. *Clearly, Theorem 3.1 ensures both the thesis of Theorem 3.1 of [10] and of Theorem 3.1 of [3].*

EXAMPLE 3.1. *The function* $F(u)=e^{-u}u^{11}+\frac{3}{5}(u+1)^{\frac{5}{3}}-\frac{3}{5}$ *satisfies the assumptions of Theorem 3.1 by choosing, for instance,* $c=1$ *and* $d=2$*. Therefore, for each* $\lambda\in]\frac{1}{8},\frac{11}{10}[$*, the problem*

$$\begin{cases} -u''=\lambda\left(e^{-u}u^{10}(11 - u) + \sqrt[3]{(u + 1)^2}\right) \\ u(0)=u(1)=0, \end{cases} \tag{3.2}$$

admits at least three non-trivial solutions in $C^2([0,1])$ *(Example 3.1 of [3]) and, moreover, for each* $h>1$*, there exists an open interval* $\Lambda\subseteq\left]0,\frac{h}{8}\right[$ *and a real positive number* σ *such that, for each* $\lambda\in\Lambda$*, the problem* (3.2) *admits at least three solutions in* $C^2([0,1])$ *whose norms in* $C^2([0,1])$ *are less than* σ*.*

We conclude this paper with the following very particular case of Theorem 3.1.

THEOREM 3.2. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a nonnegative bounded continuous function such that*

$$\int_0^1 f(t)\mathrm{d}t < \frac{1}{24}\int_0^2 f(t)\mathrm{d}t$$

Then, for each

$$\lambda\in\Lambda_1 = \left]\frac{32}{\int_0^2 f(t)dt \;-\; 2\int_0^1 f(t)dt}, \frac{2}{\int_0^1 f(t)dt}\right[,$$

the problem (3.1) *admits at least three solutions in* $C^2([0,1])$ *and, moreover, there exists an open interval*

$$\Lambda_2 \subseteq \left[0, \frac{64}{\int_0^2 f(t)dt \;-\; 16\int_0^1 f(t)dt}\right]$$

and a real positive number σ *such that, for each* $\lambda\in\Lambda_2$, *the problem* (3.1) *admits at least three solutions in* $C^2([0,1])$ *whose norms in* $C^2([0,1])$ *are less than* σ.

Proof. Taking into account that $\max_{|t|\leqslant c} F(t) = F(c)$ and $\int_0^d F(t)dt \geqslant 0$, the conclusion follows from Theorem 3.1, by choosing $c=1$, $d=2$ and $h=2$. □

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